

Synchronization of coupled nonidentical dynamical systems

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(Dated: November 24, 2011)

Abstract

We analyze the stability of synchronized state for coupled nearly identical dynamical systems on networks by deriving an approximate Master Stability Function (MSF). Using this MSF we treat the problem of designing a network having the best synchronizability properties. We find that the edges which connect nodes with a larger relative parameter mismatch are preferred and the nodes having values at one extreme of the parameter mismatch are preferred as hubs.

PACS numbers: 05.45.Xt, 05.45.-a

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When two or more dynamical systems are coupled or driven by a common signal the systems may synchronize under suitable conditions [1, 2]. One can achieve different types of synchronization, such as complete synchronization [2], phase synchronization [3], lag synchronization [4], generalized synchronization [5] etc. Recently, there is considerable interest in the synchronization of coupled dynamical systems on a network [6]. For coupled identical systems which give exact synchronization, Pecora and Carroll [7] have introduced a master stability function (MSF) which can be calculated from a simple set of master stability equations and then applied to the study of stability of the synchronous state of different networks. This general approach has become popular and has been used in various studies of synchronization on networks [8–12]. Several works on different networks have shown that small world and scale free networks show better synchronization properties [8, 9].

For coupled nonidentical systems, in general it is difficult to obtain exact synchronization. But, one can get synchronization of some generalized type [5]. The parameter mismatch between different coupled systems can lead to desynchronization bursts and this is known as the bubbling transition [13, 14]. Restrepo et al. [15] have studied the spatial patterns of such desynchronization bursts in networks. After the desynchronization burst the system returns to the synchronized state. Sun et al. [16] determine the deviation from average trajectory as a function of the mismatch. However, for nonidentical systems, there is no general theory such as MSF, to study the stability of synchronization.

In this paper we address the question of the stability of synchronization of coupled nearly identical systems on networks. By using the property of differential equations that the homogeneous part determines the exponential rates and treating the parameter mismatch in a first order perturbation theory, we derive master stability equation for coupled nearly identical systems. This master stability equation uses the homogeneous state and two parameters, α for the network coupling and Δ for the mismatch in nonidentical systems. This allows us to define the MSF and study the stability properties of the synchronized state.

When one considers identical coupled systems the important question is about the type of network which gives better synchronization properties. When one considers coupled nearly identical systems, additional interesting and important questions arise. Which nodes are better chosen as hubs? Which edges give better synchronization? Using our MSF we find that for better synchronization nodes on one extreme of parameter mismatch are preferred as hubs and nodes with larger relative parameter mismatch are preferred for constructing

edges.

Consider N coupled dynamical systems,

$$\dot{x}^i = f(x^i, r_i) + \varepsilon \sum_{j=1}^N G_{ij} h(x^j); \quad i = 1, \dots, N \quad (1)$$

where, $x^i (\in R^m)$ is an m dimensional state vector of the system i , $f : R^m \rightarrow R^m$ gives the dynamics of an isolated system, ε is a scalar coupling parameter and $h : R^m \rightarrow R^m$ is a coupling function, G is the coupling matrix of the network, r_i is some parameter which depends on the node i .

For the coupled identical systems, i.e. $r_i = r$, $\forall i$, the synchronization manifold is defined by $x^1 = \dots = x^N = x$ and is an invariant manifold provided the coupling matrix satisfies the condition that $\sum_j G_{ij} = 0$, $\forall i$. With this condition, the synchronized state x , is a solution of the uncoupled dynamics, $\dot{x} = f(x)$.

The condition $\sum_j G_{ij} = 0$ ensures that G has one eigenvector $e_1 = (1, \dots, 1)^T$, with eigenvalue $\gamma_1 = 0$. This eigenvector defines the synchronization manifold. All the remaining eigenvectors belong to the transverse manifold. The synchronized state is stable provided all the transverse Lyapunov exponents are negative.

Now, let us consider the case when the parameter r_i depends on the node i . Let the parameter mismatch be $\delta r_i = r_i - \tilde{r}$ where \tilde{r} is some typical value of the parameters r_i . In general, for nonidentical systems it is not possible to get an exact synchronization of the type discussed above. Instead we get a generalized synchronization where there is a functional relationship between variables of the systems, e.g. $g(x^i, x^j) = 0$. The generalized synchronization is stable provided the largest transverse Lyapunov exponent is negative.

To determining the stability of this generalized synchronization, we do the linear stability analysis. In this analysis, we retain terms to second order in $z^i = x^i - x$ and δr_i . The reason for doing this will be clear shortly. Thus the dynamics of the deviation z^i can be written as

$$\begin{aligned} \dot{z}^i = & D_x f(x, \tilde{r}) z^i + \varepsilon \sum_{j=1}^N G_{ij} D_x h(x) z^j + D_r f(x, \tilde{r}) \delta r_i \\ & + D_r D_x f(x, \tilde{r}) z^i \delta r_i + \frac{1}{2} D_r^2 f(x, \tilde{r}) \delta r_i^2 + \dots \end{aligned} \quad (2)$$

The terms corresponding to $(z^i)^2$ are not included since we will be interested in the solution $z^i = 0$ for finite δr_i . As an equation for z^i , the RHS of Eq. (2) contains both homogeneous and inhomogeneous terms. To a first approximation, the inhomogeneity won't affect the

exponential rate of convergence of the trajectories to the synchronous solutions though it can shift the solution. To see this consider a general linear equation $Du = p(t)$, where D is a differential operator. Let the solution be $u = u_h + g(t)$ where $u_h = \sum_i A_i h_i(t) \exp(k_i t)$ is the solution of the homogeneous equation $Du = 0$, and A_i are constants. If $p(t)$ does not have any exponential dependence, then $g(t)$ cannot contain any additional exponential other than already in u_h , due to the property that the derivative of an exponential is also an exponential with the same exponent. For example, for $\dot{u} = -ku + p$, the solution of the homogeneous equation is $u_h(t) = u(0)e^{-kt}$ and of the inhomogeneous equation with constant p is $u(t) = (u(0) - (p/k))e^{-kt} + p/k$. We note that the inhomogeneity in the differential equation shifts the asymptotic solution but does not change the exponential. In our case the stability of the synchronized state is governed by the largest transverse Lyapunov exponent, i.e. only by the exponential rates which are determined by the homogeneous equation. The inhomogeneous part will shift the solution. In addition, while calculating the Lyapunov exponents, it is necessary that the shifted solution preserves the nature of the attractor so that the average expansion and contraction rates are not significantly affected by the shift. This can be assumed to be valid when different systems are in generalized synchrony since they are related to each other. This may also hold very near the synchronization region but not far away from it.

Hence, to obtain the Lyapunov exponents, we consider the homogeneous equation obtained from Eq. (2).

$$\dot{z}^i = D_x f z^i + \varepsilon \sum_{j=1}^N G_{ij} D_x h z^j + D_r D_x f z^i \delta r_i \quad (3)$$

This equation can be put in a matrix form as [17]

$$\dot{Z} = D_x f Z + \varepsilon D_x h Z G^T + D_r D_x f Z R \quad (4)$$

where $Z = (z^1, \dots, z^N)$ is an $m \times N$ matrix and $R = \text{diag}(\delta r_1, \dots, \delta r_N)$ is an $N \times N$ diagonal matrix.

Let $\gamma_k, e_k^R, k = 1, \dots, N$ be the eigenvalues and right eigenvectors of G^T . Acting Eq. (4) on e_k^R and using the m dimensional vectors $\phi_k = Z e_k^R$, we get

$$\dot{\phi}_k = [D_x f + \varepsilon \gamma_k D_x h] \phi_k + D_r D_x f Z R e_k^R. \quad (5)$$

In general, e_k^R are not eigenvectors of R and hence Eq. (5) is not easy to treat. To solve Eq. (5) we use first order perturbation theory and write Eq. (5) as

$$\dot{\phi}_k = [D_x f + \varepsilon \gamma_k D_x u + \nu_k D_r D_x f] \phi_k \quad (6)$$

where $\nu_k = e_k^L R e_k^R$ is the first order correction and e_k^L is the left eigenvector of G^T .

Since both γ_k and ν_k can be complex, treating them as complex parameters $\alpha = \varepsilon \gamma_k$ and $\Delta = \nu_k$ respectively, we can construct the master stability equation as

$$\dot{\phi} = [D_x f + \alpha D_x h + \Delta D_r D_x f] \phi. \quad (7)$$

For the coupled identical systems, the above equation reduces to the master stability equation given by Pecora and Carroll [7]. We can determine the MSF or λ_{\max} , which is the largest Lyapunov exponent for Eq. (7), as a surface in the complex space defined by α and Δ [21]. The synchronized state is stable if the MSF is negative at each of the eigenvalues $\gamma_k = \alpha$ and $\nu_k = \Delta$ ($k \neq 1$). This ensures that all the transverse Lyapunov exponents are negative.

We note that though the master stability equation (7) uses the homogeneous state, it allows us to study the stability of the generalized synchronization in nonidentical systems. The mismatch between the different systems is included through the parameter Δ .

To examine how well Eq. (7) allows the estimation of Lyapunov exponents, we calculate the Lyapunov exponents for the coupled Rössler systems [20] and compare them with those obtained from Eq. (7). Consider N coupled chaotic Rössler systems with different frequencies,

$$\begin{aligned} \dot{x}_i &= -\omega_i y_i - z_i + \varepsilon \sum_{j=1}^N L_{ij} (x_j - x_i) \\ \dot{y}_i &= \omega_i x_i + a_r y_i \\ \dot{z}_i &= b_r + z_i (x_i - c_r) \end{aligned} \quad (8)$$

where ω_i is the frequencies of the i -th oscillator and $L_{ij} = 1$ if the nodes i and j are coupled and zero otherwise and $L_{ii} = -\sum_{j \neq i} L_{ij}$. For simplicity we restrict ourselves to symmetric coupling matrices L so that the eigenvalues and hence α and Δ are real.

We first consider two coupled Rössler oscillators. Fig. 1 plots the three largest Lyapunov exponents, $\lambda_i, i = 1, 2, 3$, as a function of the coupling strength ε and their estimated values λ_i^{MS} from Eq. (7). Fig. 2a plots the difference $\delta \lambda_i = \lambda_i - \lambda_i^{MS}$ as a function of ε for these

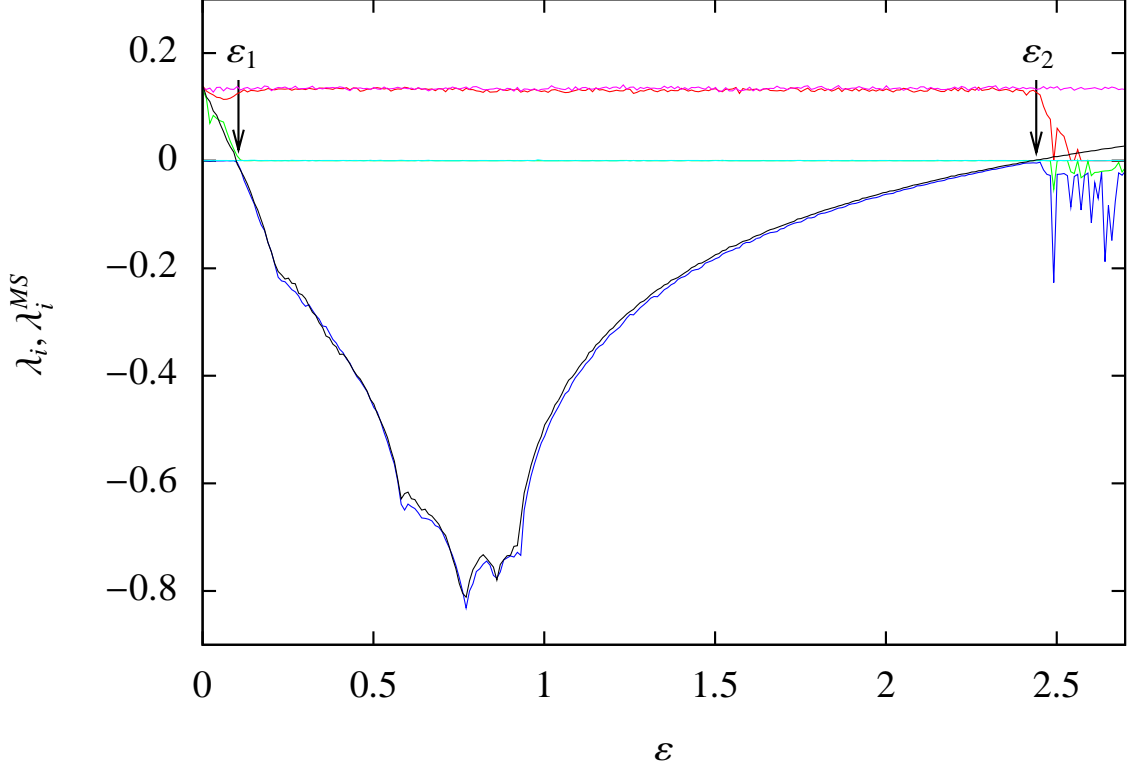


FIG. 1. The figure shows the three largest Lyapunov exponents $\lambda_i, i = 1, 2, 3$ (red, green and blue) and their estimated values λ_i^{MS} obtained from the master stability equation (Eq. (7)) (pink, cyan and black) as a function of ε for two coupled Rössler systems with frequencies $\omega_1 = 1.05$ and $\omega_2 = 1.07$. Taking $\tilde{\omega} = 1.0$ we get $\Delta_1 = \Delta_2 = 0.06$ which are used in Eq. (7). Rössler parameters are $a_r = b_r = 0.2, c_r = 7.0$. The synchronous state is stable in the region given by $\alpha_1 < \alpha < \alpha_2$ indicated by the arrows.

Lyapunov exponents. The region when the third largest Lyapunov exponent $\lambda_3 < 0$, corresponds to the synchronization region and in this region it is the largest transverse Lyapunov exponent. From Figs. 2a, we find that the differences $\delta\lambda_i$ are small in the synchronization region and very close to it. Though only three exponents are plotted in the figure, the differences are small for the other Lyapunov exponents. Fig. 2b plots the difference $\delta\lambda_i$ as a function of ε for the three largest Lyapunov exponents for a random network of sixteen nodes. Again we observe that the errors are small in the synchronization region. Thus, we find that the master stability equation (7) can estimate the actual Lyapunov exponents for the synchronized state reasonably well.

Now, we consider the MSF, λ_{max} , which is the largest transverse Lyapunov exponent.

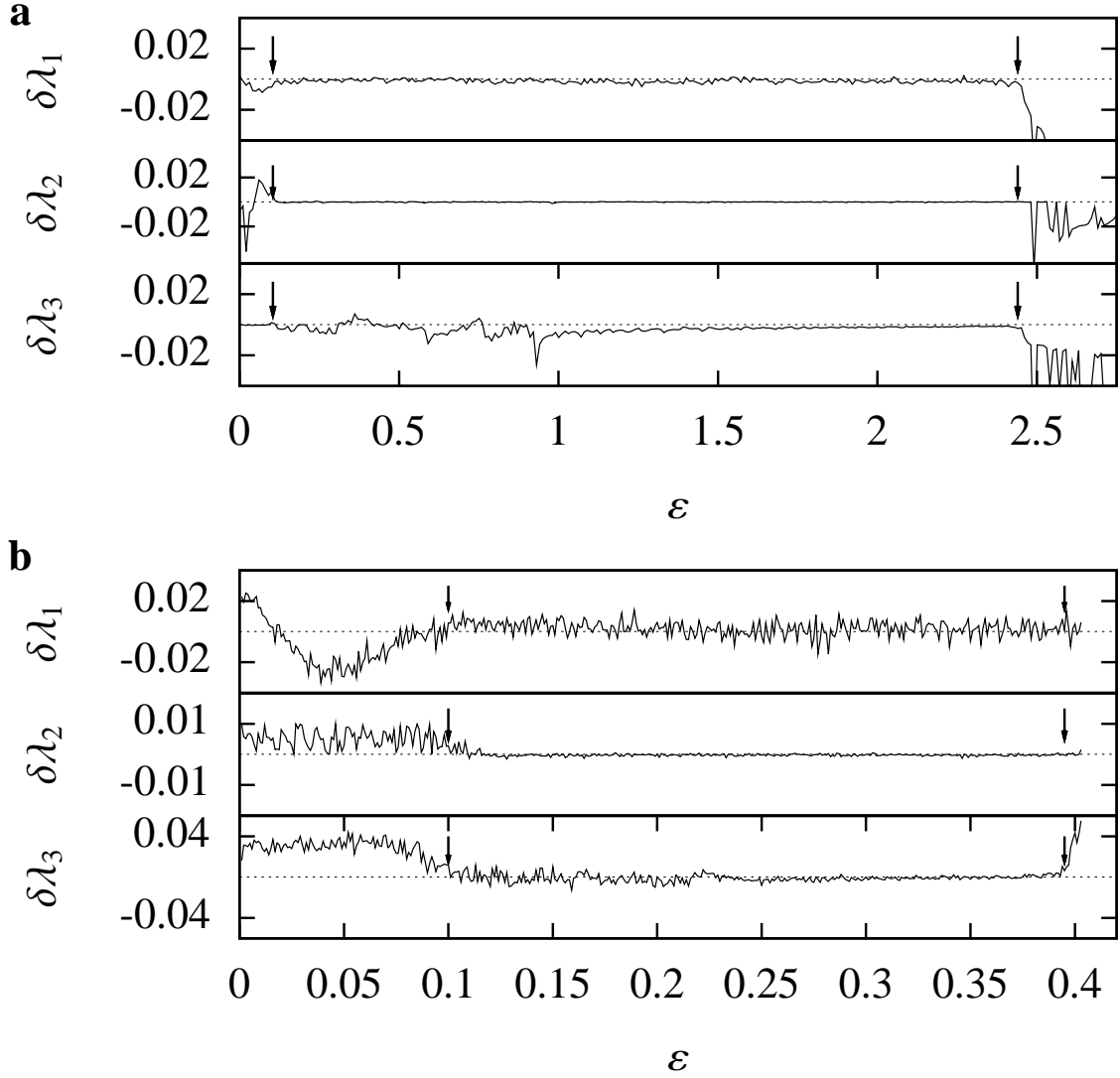


FIG. 2. **a.** The figure shows the difference $\delta\lambda_i = \lambda_i - \lambda_i^{MS}$ for the three largest Lyapunov exponents as a function of the coupling constant ε for two coupled Rössler systems with parameters as in Fig 1. **b.** The figure shows the difference $\delta\lambda_i$ for the three largest Lyapunov exponents as a function of ε for sixteen randomly coupled Rössler systems having different internal frequencies ω_i . We find that the differences are small in the synchronization region.

It can be calculated using Eq. (7). In Fig. 3 we plot λ_{max} in the parameter plane (α, Δ) as a contour plot for Rössler system. From the figure we can see that the stability region increases with the parameter Δ .

We now demonstrate the utility of the master stability function by considering the problem of construction of an optimized network which gives best synchronization properties.

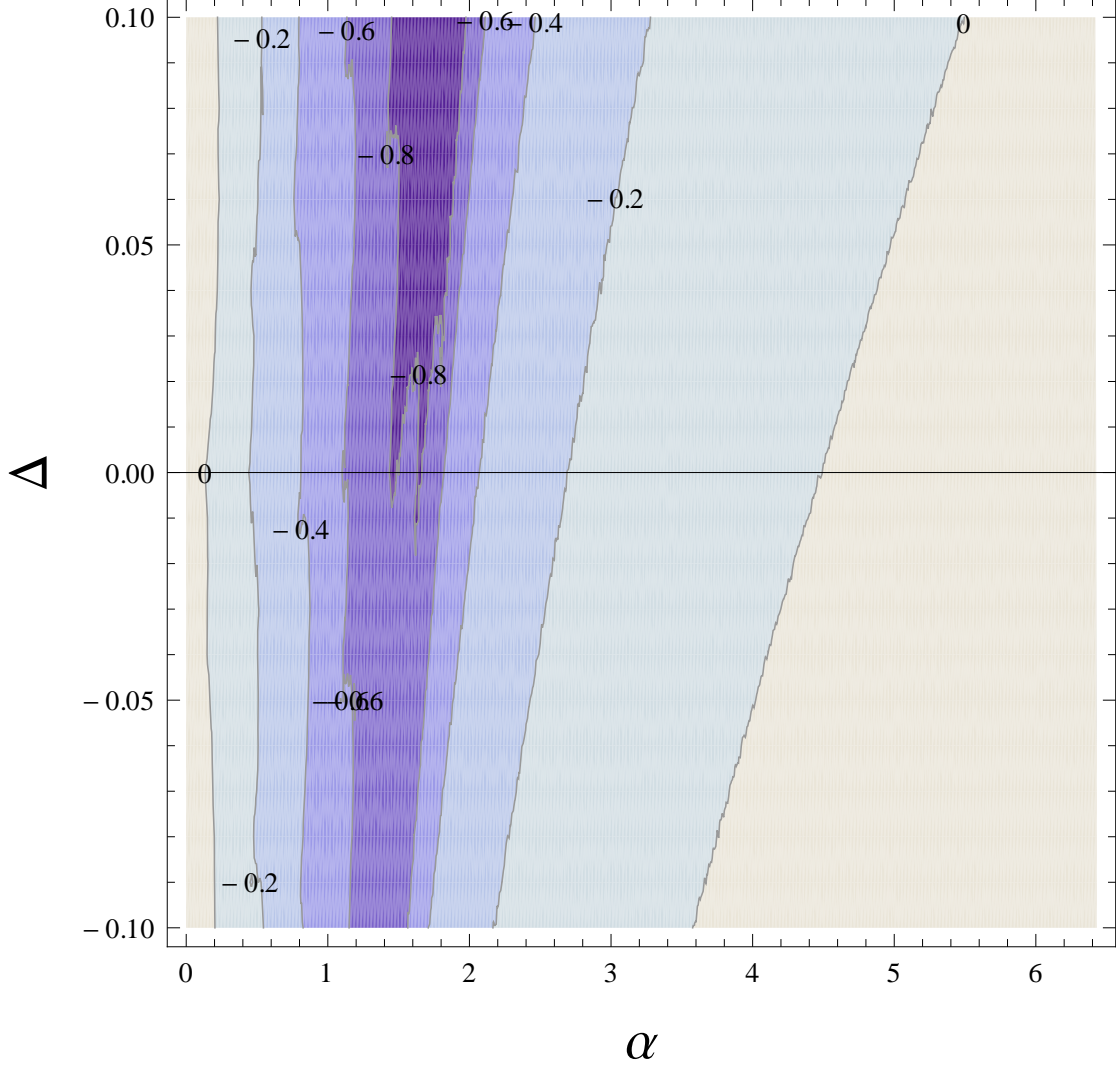


FIG. 3. The master stability function λ_{max} for Rössler system is plotted as a contour plot in the parameter plane (α, Δ) . The stability region is given by the “V” shape region bordered by the 0 contours from both sides.

To construct the optimized network we adapt Monte Carlo optimization method [22] and rewire the edges of the network to construct a network that shows best synchronizability, i.e. the largest interval l_ε of the coupling constant ε which shows synchronization.

We start with a system of nearly identical coupled Rössler oscillators as in Eq. (8) on a connected network of N nodes and E randomly chosen edges. In each Monte Carlo step we rewire one edge. If the rewired network increases the stability interval l_ε of the synchronized state, then it is chosen with probability one, otherwise it is accepted with

probability $e^{\beta(l_{\varepsilon}^{new} - l_{\varepsilon}^{old})}$ where β is the inverse temperature.

We now investigate two questions. In the optimized network, which edges are more preferable and which nodes have larger number of connection or act as hubs?

To investigate the question of which nodes act as hubs, we define the correlation coefficient between the frequency and the degree of a node as $\rho_{\omega k} = \frac{\langle (k_i - \langle k_i \rangle)(\omega_i - \langle \omega_i \rangle) \rangle}{\sqrt{\langle (k_i - \langle k_i \rangle)^2 \rangle \langle (\omega_i - \langle \omega_i \rangle)^2 \rangle}}$ where $k_i = -L_{ii}$ is the degree of node i . Fig. 4a shows $\rho_{\omega k}$ (solid line) as a function of Monte Carlo steps. For the random network $\rho_{\omega k} = 0$. We find that $\rho_{\omega k}$ increases and saturates to a positive value. Thus, in the synchronized optimized network the nodes which have larger frequencies have more connections and are preferred as hubs. The reason for this is the “V” shape of the stability region in Fig. 3, i.e. the stability range increases as Δ increases. We have also investigated a case where an opposite behavior is obtained. If instead of the frequency, we make the parameter a_r in Eq. (8) node dependent, then the stability region in the plot of MSF similar to Fig. 3, has an inverted “V” shape. In this case in the optimized network, nodes which have smaller values of a_r have more connections and are preferred as hubs.

To investigate the question of which edges are preferred, we define the correlation coefficient between the absolute frequency differences between two nodes and the edges as $\rho_{\omega a} = \frac{\langle (A_{ij} - \langle A_{ij} \rangle)(|\omega_i - \omega_j| - \langle |\omega_i - \omega_j| \rangle) \rangle}{\sqrt{\langle (A_{ij} - \langle A_{ij} \rangle)^2 \rangle \langle (|\omega_i - \omega_j| - \langle |\omega_i - \omega_j| \rangle)^2 \rangle}}$ where $A_{ij} = 1$ if nodes i and j are connected and 0 otherwise. Fig. 4b shows $\rho_{\omega a}$ as a function of Monte Carlo steps. We find that $\rho_{\omega a}$ increases from 0 (the value for the random network) and saturates. Thus, in the synchronized optimized network the pair of nodes which have a larger relative frequency mismatch are preferred as edges for the optimized network. Again, the reason for this preference of edges is probably the conical shape of the stability region in Fig. 3. The edges are to be chosen so that the parameter Δ increases and the stability region increases.

To conclude we have developed the Master Stability Function (MSF) approach for coupled nonidentical systems. We use the property of differential equations that the homogeneous part is mainly responsible for the exponential dependence of the variables. The parameter mismatch is treated in a first order perturbation theory. Our MSF uses the homogeneous state but it still allows us to study the stability properties of generalized synchronization for nonidentical systems. Using MSF, we construct optimized networks with better synchronization properties by rewiring the network keeping the number of edges constant. We find that in the optimized network the nodes having parameter mismatch at one extreme

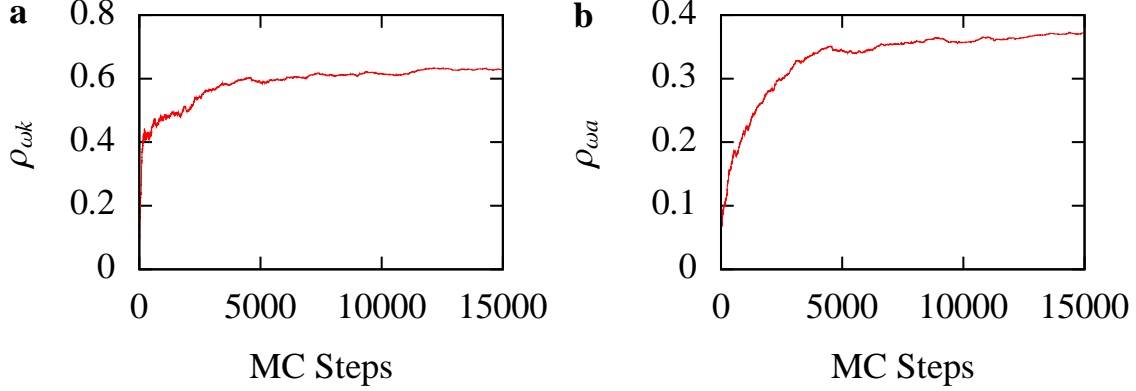


FIG. 4. The figure plots the correlation coefficient $\rho_{\omega k}$ (figure a) and $\rho_{\omega a}$ (figure b) as a function of the Monte Carlo steps of optimization for 32 coupled Rössler systems. We see that both $\rho_{\omega k}$ and $\rho_{\omega a}$ increase and saturate to positive values.

depending on the shape of stability region in MSF plot, have more edges and are preferred as hubs and the pair of nodes which have a larger relative parameter mismatch are preferred for constructing edges.

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